Problem Set Information

- Tutor: Nelson Lam (nelson.lam.lcy@gmail.com)
- Prerequisite Knowledge:
  - Frenet Serret Equation
  - Differentiation by Part
  - Orthonormal Basis & Linear Combination
- References:
  - **Lecture Notes** of Dr. LAU & Dr. CHENG
  - Thomas Calculus
- Submission & Rewards
  - Complete at least 2 harder questions during tutorial
    Reward: Any drink from vending machine
  - Complete all unfamiliar non-standard questions within 72 hours at the date of release
    WARNING : **Revising course material is probably a better way to kill time**
    Reward: A free lunch & Respect from TA, Classmates
- All the suggestions and feedback are welcome. Any report of typos is appreciated.

1 Computational Question

Exercise 1.1 ((a) 2018 TDG Quiz 2 — (b) 2020 TDG Quiz 2 — (c), (d), 2023 TDG Quiz 2)**.**

Compute the Frenet frame $\{\mathbf{T}, \mathbf{N}, \mathbf{B}\}$, curvature κ and torsion τ of the space curves below.

(a). $\alpha(\theta) = (a\cos\theta,\ a\sin\theta,\ b\theta)\ ,\ \theta \in \mathbb{R}$

(b). $\alpha(\theta) = \left(t - \sin t\cos t,\ \sin^2 t,\ \cos t\right)\ ,\ t \in (0, \pi)$

(c). $\alpha(s) = \left(\dfrac{4}{9}(1+s)^{\frac{3}{2}}, \dfrac{4}{9}(1-s)^{\frac{3}{2}}, \dfrac{1}{3}s\right), s \in (-1, 1)$

(d). $\alpha(\theta) = \left(6\cos 2\theta\cos^3\left(\dfrac{2\theta}{3}\right)\ ,\ 6\sin 2\theta\cos^3\left(\dfrac{2\theta}{3}\right)\ ,\ \dfrac{1}{2}\cos 4\theta - \cos^2 2\theta\right)\ ,\ \theta \in \left(0, \dfrac{\pi}{4}\right)$

Exercise 1.2 (2015 TDG Quiz 2)**.**
Let $\alpha : \mathbb{R} \to \mathbb{R}^3$ be a space curve defined by: $\alpha(t) = (5 + 13\cos t,\ 12 + 5\sin t,\ 13 - 12\sin t)$

(a). Compute the Frenet frame $\{\mathbf{T}, \mathbf{N}, \mathbf{B}\}$, curvature κ and torsion τ

(b). By considering $\dfrac{d}{dt}\|\alpha(t) - v(t)\|^2$, where $v : \mathbb{R} \to \mathbb{R}^3$ is a space curve. Show that α is a circle and find its centre and radius

2 Warm up Question

Exercise 2.1 (Proposition 2.4.6 & Lecture Notes Exercise 11, 16).

(a). Let $\mathbf{r}(t)$ be a regular parameterized space curve with curvature $\kappa(t) > 0$ for any t, show that $\mathbf{r}$ is a plane curve if and only if its torsion $\tau(t) = 0$ for all t

(b). Let $\mathbf{r}(t)$ be an regular arc length parameterized space curve with curvature $\kappa(s) = \kappa$ is a constant and $\tau(s) = 0$ for any s . Show that $\mathbf{r}(s)$ lies on a circle. What is the radius of the circle ?

Hints: Try to show that $\dfrac{d}{ds}\left(\mathbf{r}(s) + \dfrac{1}{\kappa}\mathbf{N}(s)\right) = 0$

Exercise 2.2 (Lecture Notes Exercise 14).
Let $\mathbf{r}(t)$ be a regular parameterized plane curve with curvature $\kappa(t) > 0$ for any t. Let $\lambda > 0$ be a constant, the parallel curve $\mathbf{r}_\lambda(t)$ is defined by:

$$\mathbf{r}_\lambda(t) \stackrel{\text{def}}{=} \mathbf{r}(t) - \lambda\,\mathbf{N}(t)$$

where $\mathbf{N}(t)$ is the unit normal vector to $\mathbf{r}$ at t. Show that the curvature of $\mathbf{r}_\lambda(t)$ is $\dfrac{\kappa}{1 + \lambda\kappa}$

Exercise 2.3 (2023 TDG Quiz 2).
Let $\mathbf{r}(t)$ be a regular parametrized space curve with $\kappa(t) > 0$ for any t. Denote the torsion at $\mathbf{r}(t)$ by $\tau(t)$. Prove that $\mathbf{r}(t)$ is contained in a plane if and only if $\tau(t) = 0$ for any t.
(Hint: A space curve $\mathbf{r}$ *is contained in a plane if there exists a fixed unit vector* $\mathbf{n}$ *such that* $\langle \mathbf{r}, \mathbf{n} \rangle$ *is a constant.)*

3 Standard Question

3.1 Lies on Sphere

Exercise 3.1 (Lecture Notes Exercise 17)**.**
Let $\alpha(s)$ be a regular space curve with arc length parameterization. $\mathbf{N}(s)$ and $\mathbf{B}(s)$ are the unit normal and unit binormal to the curve respectively. Let $\kappa(s)$ and $\tau(s)$ be the curvature and torsion of the curve. Suppose $\alpha(s)$ lies on the unit sphere for any s

(a). Prove that $\langle \alpha, \mathbf{N} \rangle = -\dfrac{1}{\kappa}$ for any s

(b). Prove that $\alpha = -\dfrac{1}{\kappa}\mathbf{N} + \dfrac{\kappa'}{\kappa^2 \tau}\mathbf{B}$ for any s

Exercise 3.2 (2021 DG AS1, 2020 TDG Final)**.** Assume that $\kappa(s) > 0$, $\tau(s) \neq 0$ and $\kappa'(s) \neq 0$ for all s for a regular curve $\alpha(s)$ parameterised by arc length. Show that α lies on a sphere if and only if:

$$\frac{1}{\kappa(s)^2} + \frac{\kappa'(s)^2}{\kappa(s)^4 \tau(s)^2} = \text{constant}$$

Exercise 3.3. Let $\alpha(s)$, where $s \in (a, b)$ be an arc-length parametrized space curve with $\kappa, \tau \neq 0$ everywhere. Show that α lies on the surface of some sphere if and only if

$$\frac{\tau(s)}{\kappa(s)} + \left(\frac{1}{\tau(s)} \left(\frac{1}{\kappa(s)} \right)' \right)' = 0.$$

3.2 Helix

Exercise 3.4 (Lecture Notes Exercise 18)**.**
Let $\alpha(s)$ be a regular space curve with arc length parameterization. $\mathbf{T}(s)$ and $\mathbf{N}(s)$ are the unit tangent vector and unit normal vector respectively. Suppose $\kappa(s) > 0$ for any s and there exists a constant C and a constant unit vector $\mathbf{u}$ such that $\langle \mathbf{T}(s), \mathbf{u} \rangle = C$ for all s

(a). Show that $\mathbf{N}(s)$ and $\mathbf{u}$ are orthogonal for all s

(b). Using (a), show that there exists a constant θ such that $\mathbf{u} = \cos\theta\,\mathbf{T}(s) + \sin\theta\,\mathbf{B}(s)$ for all s

(c). Using (b) and Frenet Serret equations, show that $\dfrac{\tau(s)}{\kappa(s)} = \cot\theta$

Exercise 3.5 (2021 DG AS1)**.** Prove that for a regular curve: $\dfrac{\kappa}{\tau} = \text{constant} \iff \langle \mathbf{T}, \mathbf{u} \rangle = \cos\theta_0$ for some $u \in \mathbb{R}^3$ and $\theta_0 \in [0, 2\pi)$

4 Harder Question

Exercise 4.1 (2014 TDG Final Exam, 2019 TDG Quiz 2).
Let $\mathbf{r}(s)$ be a unit speed curve in $\mathbb{R}^3$ with curvature $\kappa > 0$ and torsion τ, prove that the following two statements are equivalent

(1). $\exists$ fixed point $p_0 \in \mathbb{R}^3$, two functions $\lambda(s), \mu(s)$ such that $\mathbf{r}(s) - p_0 = \lambda(s)\,\mathbf{T}(s) + \mu(s)\,\mathbf{B}(s)$

(2). $\dfrac{d}{ds}\left(\dfrac{\tau(s)}{\kappa(s)}\right) =$ Constant independent of s

5 Unfamiliar Non-Standard Question

Exercise 5.1 (2023 TDG Quiz 2)**.**
Let $\alpha(s) : I \to \mathbb{R}^2$ be a regular arc length parametrized plane curve. Suppose that $\mathbf{p} \in \mathbb{R}^2$ is a point such that $\alpha(s) \neq \mathbf{p}$ for all $s \in I$. Suppose there exists $s_0 \in I$ such that

$$\|\alpha(s_0) - \mathbf{p}\| = \max_{s\in I}\{\|\alpha(s) - \mathbf{p}\|\}.$$

Denote the curvature of α at $s = s_0$ by $\kappa(s_0)$. Show that

$$|\kappa(s_0)| \geq \min_{s\in I}\left\{\frac{1}{\|\alpha(s) - \mathbf{p}\|}\right\}.$$

(**Hint 1**: recall the definition of relative extreme points in differential calculus.)
(**Hint 2**: Any relations between inner product and norm?)

Exercise 5.2 (2022 DG AS1)**.**
Let $\alpha : (a, b) \to \mathbb{R}^3$ be a regular smooth curve parameterized by arc length so that the curvature $\kappa(s) > 0$ globally. Let $s_0 \in (a, b)$. Show that for $s_3 > s_2 > s_1$ sufficiently close to s_0, $\alpha(s_1), \alpha(s_2), \alpha(s_3)$ do not collinear. prove that also as $s_i \to s_0$, the unique plane containing $\alpha(s_1), \alpha(s_2), \alpha(s_3)$ will approaches to the plane spanned by $\mathbf{T}(s_0), \mathbf{N}(s_0)$